ABSTRACTIONS OF HAMILTONIAN CONTROL SYSTEMS

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ABSTRACT. Given a control system and a desired property, an abstracted system is a reduced system that preserves the property of interest while ignoring modeling detail. In previous work, we considered abstractions of linear and analytic control systems while preserving reachability properties. In this paper we consider the abstraction problem for Hamiltonian control systems, and abstract systems while preserving their Hamiltonian structure. We show how the mechanical structure of Hamiltonian control systems can be exploited in the abstraction process. We then focus on local accessibility preserving abstractions and provide conditions under which local accessibility properties of the abstracted Hamiltonian system are equivalent to the accessibility properties of the original Hamiltonian control system.

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1. Introduction

Abstractions of control systems are important for reducing the complexity of their analysis or design. From an analysis perspective, given a large-scale control system and a property to be verified, one extracts a smaller *abstracted* system with equivalent properties. Checking the property on the abstraction is then equivalent to checking the property on the original system. From a design perspective, rather than designing a controller for the original large scale system, one designs a controller for the smaller abstracted system, and then refines the design to the original system while incorporating modeling detail.

A formal approach to a modeling framework of abstraction critically depends on whether we are able to *construct* hierarchies of abstractions as well as characterize conditions under which various properties propagate from the original to the abstracted system and vice versa. In [8], hierarchical abstractions of linear control systems were extracted using computationally efficient constructions, and conditions under which controllability of the abstracted system implied controllability of the original system were obtained. This led to extremely efficient hierarchical controllability algorithms.

In the same spirit, abstractions of analytic control systems were considered in [13]. The canonical construction for linear systems was generalized to analytic systems, yielding a canonical construction for extracting abstractions of nonlinear control systems. The condition under which local accessibility of the abstracted system is equivalent to the local accessibility of the original system captured the linear condition of [12].

In this paper, we proceed in the spirit of [13] and consider abstractions of Hamiltonian control systems. In Hamiltonian control systems are completely specified by controlled Hamiltonians. This additional structure allow a simplification of the abstraction process since the relevant information that must be captured by the

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abstracted system is simply the controlled Hamiltonian. On the other hand, to be able to relate the dynamics induced by the controlled Hamiltonians we need to restrict the class of abstracting maps to those that preserve the Hamiltonian structure.

Given a Hamiltonian control system on Poisson manifold M, and a (quotient) Poisson map $\phi : M \longrightarrow N$, we present a canonical construction that extracts an abstracted Hamiltonian system on N. This canonical construction is dual to the construction of [13]. We then characterize abstracting maps for which the original and abstracted system are equivalent from a local accessibility point of view.

Reduction of mechanical control systems is a very rich and mature area [5, 9, 7, 10]. The approach presented in this paper is quite different from these established notions of reduction for mechanical systems. When performing an abstraction one is interested in ignoring irrelevant modeling details. In this spirit one quotients the original model by groups actions that do not necessarily represent symmetries. This extra freedom in performing reduction is balanced by the fact that information about the system is lost when performing an abstraction, whereas when reducing using symmetries no essential information is lost. However abstracting a control system and in particular an Hamiltonian one is always possible therefore leading to a more general notion of reduction.

The structure of this paper is as follows : In Section 2 we review Poisson geometry in order to establish notation. In Section 3 we present a global definition of Hamiltonian control systems, and in Section 4 we define abstractions of Hamiltonian control systems. In Section 5 we obtain a canonical construction for abstracting Hamiltonian control systems, and characterize local accessibility equivalence between the original and the abstracted system. Section 6 illustrated our results on a spherical pendulum example, and Section 7 points to interesting future research.

2. Mathematical Preliminaries

In this section we review some basic facts from differential and Poisson geometry as well as control theory and Hamiltonian control systems, in order to establish consistent notation. The reader may whish to consult numerous books on these subjects such as [1, 2, 11, 6].

2.1. Differential Geometry.

Let M be a differentiable manifold and T_xM its tangent space at $x \in M$. The tangent bundle of M is denoted by $TM = \cup_{x\in M} T_xM$ and π is the canonical projection map $\pi : TM \longrightarrow M$ taking a tangent vector $X(x) \in T_xM \subset TM$ to the base point $x \in M$. Dually we define the cotangent bundle as $T^*M = \cup_{x\in M} T_x^*M$, where T_x^*M is the cotangent space of M at x. Now let M and N be smooth manifolds and $\phi : M \longrightarrow N$ a smooth map. Given a map $\phi : M \longrightarrow N$, we denote by $T_x\phi : T_xM \longrightarrow T_{\phi(x)}N$ the induced tangent which maps tangent vectors from T_xM to tangent vectors at $T_{\phi(x)}N$.

A fiber bundle is a tuple $(B, M, \pi_B, U, \{O_i\}_{i\in I})$, where B, M and U are smooth manifolds called the *total space*, the *base space* and *standard fiber* respectively. The map $\pi_B : B \longrightarrow M$ is a surjective submersion and $\{O_i\}_{i\in I}$ is an open cover of M such that for every $i \in I$ there exists a diffeomorphism $\Psi_i : \pi_B^{-1}(O_i) \longrightarrow O_i \times U$ satisfying $\pi_o \circ \Psi_i = \pi_B$, where π_o is the projection from $O_i \times U$ to O_i. The submanifold $\pi^{-1}(x)$ is called the fiber at $x \in M$.

2.2. **Poisson Geometry.** For the purposes of this paper, it will be more natural to work with Poisson manifolds, rather than symplectic manifolds [1]. A Poisson structure on manifold M is a bilinear map from $C^\infty(M) \times C^\infty(M)$ to $C^\infty(M)$ called *Poisson bracket*, denoted by $\{f,g\}_M$ or simply $\{f,g\}$, satisfying the following identities

$$\{f,g\} = -\{g,f\} \qquad \text{skew-symmetry} \tag{2.1}$$

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \qquad \text{Jacobi identity} \tag{2.2}$$

$$\{f,gh\} = \{f,g\}h + g\{f,h\} \qquad \text{Leibnitz rule} \tag{2.3}$$

A Poisson manifold $(M,\{,\}_M)$ is a smooth manifold M equipped with a Poisson structure $\{,\}_M$. Given a smooth function $h : M \longrightarrow \mathbb{R}$, the Poisson bracket allows us to obtain a Hamiltonian vector field X_h with Hamiltonian h using

$$\mathcal{L}_{X_h} f = \{f,h\} \qquad \forall f \in C^\infty(M) \tag{2.4}$$

where $\mathcal{L}_{X_h} f$ is the Lie derivative of f along X_h. Note that the vector field X_h is well defined since the Poisson bracket verifies the Leibnitz rule and therefore defines a derivation on $C^\infty(M)$ ([10]). Furthermore $C^\infty(M)$ equipped with $\{,\}$ is a Lie algebra, also called a Poisson algebra. Associated with the Poisson bracket there is a contravariant anti-symmetric two-tensor

$$B : T^*M \times T^*M \longrightarrow \mathbb{R} \tag{2.5}$$

such that

$$B(x)(\mathrm{d}f, \mathrm{d}g) = \{f,g\}(x) \tag{2.6}$$

We say that the Poisson structure is non-degenerate if the map $B^\# : T^*M \longrightarrow TM$ defined by

$$\mathrm{d}g(B^\#(\mathrm{d}f)) = B(\mathrm{d}f, \mathrm{d}g)$$

is an isomorphism for every $x \in M$. Given a map $\phi : (M,\{,\}_M) \longrightarrow (N,\{,\}_N)$ between Poisson manifolds, we say that ϕ preserves the Poisson structure or that ϕ is a Poisson map iff

$$\{f\circ\phi, g\circ\phi\}_M = \{f,g\}_N \circ \phi \tag{2.7}$$

for every $f,g \in C^\infty(N)$. The classical Hamilton equations can be recovered using the Poisson bracket. Let N be any manifold of dimension n, then $M = T^*N$ is a Poisson manifold of dimension $2n$ with natural coordinates given by (q_i, p_i). The canonical Poisson bracket is

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \tag{2.8}$$

Given a smooth map $h : M \longrightarrow \mathbb{R}$ the Hamiltonian vector field X_h is given in the natural coordinates by:

$$\frac{d}{dt}q^i = \mathcal{L}_{X_h} q^i = \{q^i, h\} = \frac{\partial h}{\partial p_i} \tag{2.9}$$

$$\frac{d}{dt}p_i = \mathcal{L}_{X_h} p_i = \{p_i, h\} = -\frac{\partial h}{\partial q^i} \tag{2.10}$$

which is just the usual form of Hamilton equations.

[1] More detailed expositions on symplectic and Poisson geometry can be found in [10, 3]

3. Hamiltonian Control Systems

Before defining Hamiltonian control systems, we present a global definition of a control systems [11].

Definition 3.1 (Control System)**.** A control system $S = (U, F)$ consists of a fiber bundle $\pi : U \longrightarrow M$ called the control bundle and a smooth map $F : U \longrightarrow TM$ which is fiber preserving, that is $\pi' \circ F = \pi$ where $\pi' : TM \longrightarrow M$ is the tangent bundle projection. Given a control system $S = (U, F)$, the control distribution $\mathcal{D}$ of control system S, is naturally defined pointwise by $\mathcal{D}(x) = F(\pi^{-1}(x))$ for all $x \in M$.

The control space U is modeled as a fiber bundle since in general the control inputs available may depend on the current state of the system. In local coordinates, Definition 3.1 reduces to the familiar $\dot{x} = f(x, u)$ with $u \in \pi^{-1}(x)$. Using this definition of control systems, the concept of trajectories of control systems becomes as follows.

Definition 3.2 (Trajectories of Control Systems)**.** A curve $c : I \longrightarrow M$, $I \subseteq \mathbb{R}_0^+$ is called an *trajectory* of control system $S = (U, F)$, if there exists a curve $c^U : I \longrightarrow U$ satisfying:

$$\begin{aligned} \pi \circ c^U &= c \\ \frac{d}{dt}c(t) &= F(c^U) \end{aligned}$$

Again in local coordinates, the above definition says that $x(t)$ is a trajectory of a control system if there exists an input $u(t)$ such that $x(t)$ satisfies $\dot{x}(t) = f(x(t), u(t))$ and $u(t) \in U(x(t)) = \pi^{-1}(x(t))$ for all $t \in I$.

Hamiltonian control systems are control systems endowed with additional structure. The extra structure comes from the fact that they model mechanical systems so they are essentially a collection of Hamiltonian vector fields parameterized by the control input. The following global and coordinate free description of Hamiltonian control systems is inspired from [14].

Definition 3.3 (Hamiltonian Control Systems)**.** A Hamiltonian control system $S_H = (U, H)$ consists of a control bundle $\pi : U \longrightarrow M$ over a Poisson manifold $(M, \{,\})$ with non-degenerate Poisson bracket, and a smooth function $H : U \longrightarrow \mathbb{R}$. With the Hamiltonian control system $S_H = (U, H)$ we associate the collection of Hamiltonian $\mathcal{H}$ as the collection of all smooth functions satisfying $\mathcal{H}(x) = H(\pi^{-1}(x))$ for all $x \in M$. This family induces the control distribution $\mathcal{D}_H$ defined pointwise by $\mathcal{D}_H(x) = X_{\mathcal{H}(x)}$, where for all $x \in M$, $X_{\mathcal{H}(x)}$ satisfies $\mathcal{L}_{X_{\mathcal{H}(x)}} f = \{f, \mathcal{H}\}(x)$, that is $\mathcal{L}_{X_{\mathcal{H}(x)}} f = \{f, h\}(x)$ for all $h \in \mathcal{H}(x)$, $f \in C^\infty(M)$.

The map H should be thought of as a controlled Hamiltonian since it assigns a Hamiltonian function to each control input. Note that the control bundle, and the controlled Hamiltonian completely specify the Hamiltonian control system. In particular, by fixing the control input, one obtains a Hamiltonian vector field.

4. Abstractions of Hamiltonian Control Systems

Given a Hamiltonian control system[2] S_{H_M} defined on a Poisson manifold $(M, \{,\}_M)$ our goal is to construct a map $\phi : M \longrightarrow N$, the *abstraction map* or *aggregation map* that will induce a new Hamiltonian control system S_{H_N} on the lower dimensional Poisson manifold $(N, \{,\}_N)$ having as trajectories $\phi(c^M)$, where c^M are S_{H_M}

[2]From now on, $S_{H_M} = (U_M, H_M)$ or simply S_{H_M} denotes a Hamiltonian control system on Poisson manifold $(M, \{,\}_M)$.

trajectories. The concept of abstraction map for continuous, not necessarily Hamiltonian, control systems is defined in [8].

Definition 4.1 (Abstracting Maps)**.** Let S_M and S_N be two control systems on manifolds M and N, respectively. A smooth surjective submersion $\phi : M \longrightarrow N$ is called an abstraction or aggregation map iff for every trajectory c^M of S_M, $\phi(c^M)$ is a trajectory of S_N. Control system S_N is called a ϕ-abstraction of S_M.

From the above definition it is clear that an abstraction captures all the trajectories of the original system, but may also contain redundant trajectories. These redundant trajectories are not feasible by the original system and are therefore undesired. Clearly, it is difficult to determine whether a control system is an abstraction of another at the level of trajectories. One is then interested in a characterization of abstractions which is equivalent to Definition 4.1 but checkable. To leads to the notion of ϕ-related control systems.

Definition 4.2 (ϕ-related control systems [8])**.** Let S_M and S_N be two control systems defined on manifolds M and N, respectively. Let $\phi : M \longrightarrow N$ be a surjective submersion. Then control systems S_M and S_N are ϕ-related iff for every $x \in M$:

$$T_x\phi\left(\mathcal{D}_M(x)\right) \subseteq \mathcal{D}_N(\phi(x)) \tag{4.1}$$

The notion of ϕ-related control system is a generalization of the notion of ϕ-related vector fields commonly found in differential geometry. It is also evident that given a control system S_M, there is a *minimal* ϕ-related control system S_N, up to control parameterization. The relationship between ϕ-abstractions and ϕ-related control systems is now given.

Theorem 4.3 ([12])**.** *Let S_M and S_N be control systems on manifolds M and N, respectively, and $\phi : M \longrightarrow N$ a smooth map. Then S_M and S_N are ϕ-related control systems if and only if S_N is a ϕ-abstraction of S_M.*

We now consider these notions for Hamiltonian control systems. Since Hamiltonian control systems are uniquely determined by their controlled Hamiltonian, the notion of ϕ-related control systems specializes to Hamiltonian control systems as follows:

Definition 4.4 (ϕ-related Hamiltonian control systems)**.** Let S_{H_M} and S_{H_N} be two Hamiltonian control systems defined on Poisson manifolds $(M, \{,\}_M)$ and $(N, \{,\}_N)$, respectively. Let $\phi : M \longrightarrow N$ be a surjective Poisson submersion, and let ϕ_B be defined by $\phi_B = (B_N^{\#})^{-1} \circ T\phi \circ B_M^{\#}$. Then Hamiltonian control systems S_{H_M} and S_{H_N} are ϕ-related iff for all $x \in M$,

$$\phi_B(\mathrm{d}\mathcal{H}_M(x)) \subseteq \mathrm{d}\mathcal{H}_N(\phi(x)) \tag{4.2}$$

Although the above definition is stated in terms of the exterior derivative of the family of Hamiltonian defining the control systems, a canonical construction to be presented at section 4.8 will allows us to compute $\mathcal{H}_N$ directly from from $\mathcal{H}_M$. The relation between ϕ-related Hamiltonian control systems and ϕ-abstractions parallels the general case.

Proposition 4.5. *Let S_{H_M} and S_{H_N} be Hamiltonian control systems on Poisson manifolds $(M, \{,\}_M)$ and $(N, \{,\}_N)$, respectively, and $\phi : M \longrightarrow N$ a smooth Poisson map. Then S_{H_M} and S_{H_N} are ϕ-related if and only if S_{H_N} is a ϕ-abstraction of S_{H_M}.*

Proof. It is enough to show that if ϕ is a Poisson map then Definitions 4.2 and 4.4 are equivalent for Hamiltonian control systems. The result then follows from Theorem 4.3.

Definition 4.4 is equivalent to:

$$\begin{aligned} \phi_B(\mathrm{d}\mathcal{H}_M(x)) &\subseteq \mathrm{d}\mathcal{H}_N(\phi(x)) \Leftrightarrow \\ T_x\phi(B_M^\#(\mathrm{d}\mathcal{H}_M(x))) &\subseteq B_N^\#(\mathrm{d}\mathcal{H}_N(\phi(x))) \Leftrightarrow \\ T_x\phi(\mathcal{D}_{H_M}(x)) &\subseteq \mathcal{D}_{H_N}(\phi(x)) \end{aligned}$$

which is just Definition 4.2. □

Proposition 4.5 tell us that the abstracting process can be characterized at the level of the controlled Hamiltonians. This result should be expected since the controlled Hamiltonians completely specify the dynamics of Hamiltonian control systems given a Poisson structure.

4.1. **Constructing Poisson maps.** In order to extract a Hamiltonian abstraction from an Hamiltonian control system S_{H_M} on a Poisson manifold $(M, \{,\}_M)$, one needs a Poisson map $\phi : M \longrightarrow N$ that will induce the abstraction on N. In many cases, however, one only knows which variables are unimportant and which should be ignored. How should this information should be assembled to define an abstracting Poisson map? We must ensure that (1) ϕ is a Poisson map, and (2) the Poisson bracket $\{,\}_N$ is non-degenerate. Even if ϕ is Poisson and $dim(N)$ is even it is not true, in general, that the bracket in N is non-degenerate as the following example shows.

Let $M = T^*\mathbb{R}^3$ with the canonical bracket, that is $\{f, g\}_M = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$. Denote a point in M by $x = (q^1, q^2, q^3, p_1, p_2, p_3)$ and let $\phi(q^1, q^2, q^3, p_1, p_2, p_3) = (q^1, q^2, q^3, p^1)$. The map ϕ is Poisson as can easily be verified but the bracket induced on N and given by

$$\{f, g\}_N = \frac{\partial f}{\partial q^1}\frac{\partial g}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q^1} \tag{4.3}$$

is degenerate since its rank is only 2. This example also shows that to avoid these problems one must make sure that the directions collapsed by ϕ are conjugate. More precisely we have the following well known result,

Proposition 4.6. *Let $(M, \{,\}_M)$ be a Poisson manifold with non-degenerate Poisson bracket and $\phi : M \longrightarrow N$ an abstracting map. If for every $X \in Ker(T\phi)$, X is Hamiltonian with Hamiltonian function $h \in C^\infty(M)$ and there exists a $g \in C^\infty(M)$ such that $\{h, g\}_M \neq 0$ and $X_g \in Ker(T\phi)$ then ϕ is Poisson and induces a non-degenerate Poisson bracket on N by:*

$$\{f_1, f_2\}_N \circ \phi = \{f_1 \circ \phi, f_2 \circ \phi\}_M \tag{4.4}$$

Proof. Since the map ϕ is a surjective submersion it defines a regular equivalence relation $=_\phi$ by declaring two points x and x' to be on the same equivalence class iff $\phi(x) = \phi(x')$. The equivalence classes of this relation are described by the orbits of $Ker(T\phi)$. Since every element of $Ker(T\phi)$ is Hamiltonian the orbit of $Ker(T\phi)$ is an Hamiltonian action of $\mathbb{R}^k$ with $k = dim(Ker(T\phi))$. The quotient manifold $M/ =_\phi$ which is diffeomorphic to N inherits a Poisson structure defined by (4.4), see for example [9, 10].

We will only show that $\{,\}_N$ is non-degenerate. By Lie-Weinstein theorem [15] there is a local coordinate transformation $\varphi_M : M \longrightarrow M$ such that in the new canonical coordinates $(q_1, q_2, \ldots, q_m, p_1, p_2, \ldots, p_m, c_1, c_2 \ldots, c_v)$ the following holds $\{q_i, q_j\}_M = \{p_i, p_j\}_M = \{q_i, c_j\}_M = \{p_i, c_j\}_M = \{c_i, c_j\}_M = 0$ and $\{q_i, p_j\}_M = \delta_{ij}$. Since

$\{,\}_M$ is non degenerate the coordinates reduce to $(q_1, q_2, \ldots, q_m, p_1, p_2, \ldots, p_m)$. By assumption for every h such that $X_h \in Ker(T\phi)$ there is an g such that $\{h, g\}_M \neq 0$, and we can write $\{h \circ \varphi_M^{-1}, g \circ \varphi_M^{-1}\}_M = \{h, g\}_M \circ \varphi_M^{-1}$. Since $\{h, g\}_M \neq 0$ it follows that $\{h \circ \varphi_M^{-1}, g \circ \varphi_M^{-1}\}_M \neq 0$ meaning that $h = q_i$ and $g = p_i$ for some i. We can assume without loss of generality, that the new canonical coordinates $q_1, q_2, \ldots, q_n$ and $p_1, p_2, \ldots, p_n$ are such that X_{q_i} and X_{p_i} belong to $Ker(T\phi)$ for $i = 1 \ldots n$. Consider then the map $\phi = \varphi_M^{-1} \circ \phi \circ \varphi_N : M \longrightarrow N$ defined on a open set around the point $x \in M$. This map sends $(q_1, \ldots, q_m, p_1, \ldots, p_m)$ to $(q_{n+1}, q_{n+2}, \ldots, q_m, p_{n+1}, p_{n+2}, \ldots, p_m)$ and therefore $\{,\}_N$ is nondegenerate at x. Since this holds for any $x \in M$ N is non degenerate Poisson manifold. □

To use Theorem 4.3 the Hamiltonian maps for $Ker(T\phi)$ need to belong to $\mathcal{P}(\mathcal{H}_M)$ so we have the following construction to build an abstracting map ϕ. Pick a collection of maps $h_1, h_2, \ldots, h_n \in C^\infty(M)$ such that $h_i \in \mathcal{P}(\mathcal{H}_M)$ and determine the conjugate to h_i, that is a map h_i^c such that $\{h_i, h_i^c\}_M \neq 0$. If h_i^c also belongs to $\mathcal{P}(\mathcal{H}_M)$ then any map ϕ such that $Ker(T\phi) = Span\{X_{h_1}, X_{h_1^c}, X_{h_2}, X_{h_2^c}, \ldots, X_{h_n}, X_{h_n^c}\}$ verifies the conditions of Proposition 4.6. A illustration of this construction can be found in Section 6.

4.2. **Canonical Construction.** Given a Poisson map, Definition 4.4 provides us with a geometric definition for Hamiltonian abstractions which is useful conceptually but not computationally. We now present a canonical construction that will allow us to obtain an abstraction S_{H_N} from an Hamiltonian control system S_{H_M} and a Poisson map $\phi : M \longrightarrow N$. Our construction is inspired from the canonical construction of [13], even though the construction presented here uses codistributions as opposed to distributions. This is natural for Hamiltonian systems since the differentials of the Hamiltonians capture all system information.

Definition 4.4 and, in particular, condition (4.2) require the union of all the values of $\mathrm{d}\mathcal{H}_M$ evaluated at any $x \in \phi^{-1}(y)$. A way of constructing this union is to define another family of maps $\mathcal{F}$ such that $\mathrm{d}\mathcal{F}$ is *constant* along $\phi^{-1}(y)$ and furthermore satisfies $\mathrm{d}\mathcal{H}_M \subseteq \mathrm{d}\mathcal{F}$. From this new family it suffices to compute $\mathrm{d}\mathcal{H}_N(y) = \mathrm{d}\mathcal{F}(x)$ for some $x \in \phi^{-1}(y)$ since $\mathrm{d}\mathcal{F}$ is the same for any $x \in \phi^{-1}(y)$. In other words, we would like to construct a family of maps $\mathcal{F}$ such that

1. $\mathrm{d}\mathcal{H}_M \subseteq \mathrm{d}\mathcal{F}$
2. For all $x, x' \in M$ such that $\phi(x) = \phi(x')$, $\mathrm{d}\mathcal{F}(x) = \mathrm{d}\mathcal{F}(x')$.

Let $\mathcal{K}$ be the integrable distribution $Ker(T\phi)$. Then the leaves of the foliation $\mathcal{K}$ correspond to points on M that have the same image under ϕ. In this setting, we would like to design the family $\mathcal{F}$ so that the resulting codistribution $\mathrm{d}\mathcal{F}$ is invariant with respect to the vector fields in K. This idea is captured in the following proposition.

Proposition 4.7 (Invariant Codistributions). *A collection $\mathcal{F}$ of smooth functions satisfies $\mathrm{d}\mathcal{F}(x) = \mathrm{d}\mathcal{F}(x')$ for all $x, x' \in M$ such that $\phi(x) = \phi(x')$ if and only if $\mathcal{L}_K \mathrm{d}f \in \mathrm{d}\mathcal{F}$ for all $K \in \mathcal{K}$ and all maps $f \in \mathcal{F}$.*

Proof. Instead of working with the one-forms $\mathrm{d}f_1$, $\mathrm{d}f_2$, $\ldots$, $\mathrm{d}f_v$, spanning $\mathrm{d}\mathcal{F}$ we can associate the v-form $\alpha = \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \ldots \wedge \mathrm{d}f_v$ with the vector space spanned by these forms since any other set of one-forms $\{\beta_1, \beta_2, \ldots, \beta_v\}$ spanning the same vector space verify $\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_v = \lambda\alpha$ for some smooth function λ.

We first show that if $\mathrm{d}\mathcal{F}(x) = \mathrm{d}\mathcal{F}(x')$ for $\phi(x) = \phi(x')$ then $\mathcal{L}_K \mathrm{d}f \in \mathrm{d}\mathcal{F}$. Let $\Phi_t(x)$ be the integral curve of some vector field K belonging to $\mathcal{K}$ satisfying $\Phi_0(x) = x$. The equality $\mathrm{d}\mathcal{F}(x) = \mathrm{d}\mathcal{F}(x')$ can be replaced by

$\mathrm{d}\mathcal{F}(x) = \mathrm{d}\mathcal{F}(\Phi_t(x))$ for all $t \in \mathbb{R}$. Differentiating with respect to time we get $\mathcal{L}_K\alpha = 0$ which is equal to

$$\begin{aligned} \mathcal{L}_K\alpha &= \mathrm{d}(\imath_K\alpha) + \imath_K(\mathrm{dd}\alpha) \\ &= \mathrm{d}\alpha(K) \end{aligned}$$

Computing $\mathrm{d}\alpha(K)$ in coordinates and equating to zero we get that

$$\mathrm{d}(\mathrm{d}f_i(K)) = 0 \quad \text{or} \quad \mathrm{d}(\mathrm{d}f_i(K)) = a_j\mathrm{d}f_j \tag{4.5}$$

which simply means that $\mathrm{d}\mathcal{L}_K f_i = \mathcal{L}_K\mathrm{d}f_i = a_j\mathrm{d}f_j$ from which we conclude that $\mathcal{L}_K\mathrm{d}f \in \mathrm{d}\mathcal{F}$ for every $f \in \mathcal{F}$. For the converse assume that $\mathcal{L}_K\mathrm{d}f \in \mathrm{d}\mathcal{F}$. We want to show that

$$\alpha(x) = \alpha(\Phi_t) \quad \text{for all } t \in \mathbb{R} \tag{4.6}$$

But the derivative of (4.6) is true by assumption and (4.6) holds for $t = 0$. ☐

Proposition 4.7 motivates a canonical constructive procedure to obtain the abstracted hamiltonian control system S_{H_N} given an hamiltonian control system S_{H_M} and an abstracting Poisson map $\phi : M \longrightarrow N$. If we denote the anihilating codistribution of $\mathcal{K}$ by $\mathcal{K}^o$, that is $\mathcal{K}^o = \{\beta \in T^*M \quad | \quad \beta(K) = 0 \quad \forall K \in \mathcal{K}\}$ we can construct a collection of Hamiltonians $\mathcal{H}_N$ based on $\mathcal{H}_M$ as follows:

Definition 4.8 (Canonical construction). Let $\phi : (M, \{,\}_M) \longrightarrow (N, \{,\}_N)$ be a Poisson map between manifolds with non-degenerate Poisson brackets, and let $\mathcal{H}_M$ be a collection of Hamiltonians on M. Denote by $\overline{\mathcal{H}}_M$ the following family of smooth maps:

$$\overline{\mathcal{H}}_M = \mathcal{H}_M \cup \mathcal{L}_K\mathcal{H}_M \cup \mathcal{L}_K\mathcal{L}_K\mathcal{H}_M \cup \ldots \tag{4.7}$$

for all $K \in \mathcal{K}$. The collection of Hamiltonians $\mathcal{H}_N$ defined by

$$\mathcal{H}_N = \overline{\mathcal{H}}_M \circ i \tag{4.8}$$

for any embedding $i : N \hookrightarrow M$ such that $B_M^\#(\mathcal{K}^o) \subseteq Ti(TN)$ is called *canonically* ϕ-related to $\mathcal{H}_M$.

The collection of Hamiltonians obtained by construction of Definition 4.8 is canonical in the following sense.

Proposition 4.9 (Minimal Abstraction). *The codistribution* $\mathrm{d}\overline{\mathcal{H}}_M$ *is the smallest codistribution satisfying*

1. $\mathrm{d}\mathcal{H}_M \subseteq \mathrm{d}\overline{\mathcal{H}}_M$
2. *For all* $x_1, x_2 \in M$ *such that* $\phi(x_1) = \phi(x_2)$, $\mathrm{d}\overline{\mathcal{H}}_M(x_1) = \mathrm{d}\overline{\mathcal{H}}_M(x_2)$.

and the Hamiltonian control system defined by $\mathcal{H}_N$ *is the smallest Hamiltonian control system* ϕ*-related to* $\mathcal{H}_M$.

Proof. The collection of Hamiltonians $\overline{\mathcal{H}}_M$ contains $\mathcal{H}_M$ by construction so that is follows trivially that it satisfies property 1. It also verifies property 2 since by construction it satisfies the conditions of Proposition 4.7. To show that this codistribution is the smallest one verifying these properties consider any other collection of Hamiltonians $\mathcal{G}_M$ also verifying properties 1 and 2. Let h be any map such that $h \in \overline{\mathcal{H}}_M$, if $\mathrm{d}h \in \mathrm{d}\mathcal{H}_M$ then $\mathrm{d}h \in \mathrm{d}\mathcal{G}_M$ because $\mathcal{G}_M$ verifies property 1. Otherwise $\mathrm{d}h$ is obtained from a finite number of Lie derivatives along K of some one-form in $\mathrm{d}\mathcal{H}_M$. Since $\mathcal{L}_K\mathrm{d}f = \mathrm{d}\mathcal{L}_K f$, property 2 and Proposition 4.7 imply that $\mathrm{d}h \in \mathrm{d}\mathcal{G}_M$. The above discussion shows that for all $\mathrm{d}h \in \mathrm{d}\overline{\mathcal{H}}_M$ we have that $\mathrm{d}h \in \mathrm{d}\mathcal{G}_M$, therefore $\mathrm{d}\overline{\mathcal{H}}_M$ is contained in any other codistribution verifying properties 1 and 2.

To prove the second assertion we need to show first that the Hamiltonian control system defined by $\mathcal{H}_N$ is ϕ-related to S_{H_M}. To see that this is the case consider any $h_M \in \mathcal{H}_M$ and any point $x \in M$. By definition of $\mathcal{H}_N$ there is a map h_N satisfying $h_N = h_M \circ i$, for some i embedding N into M through the point x. The Hamiltonian h_N defines a Hamiltonian in M by $h_N \circ \phi$ and since ϕ is Poisson we have that $T\phi \cdot X_{h_N \circ \phi} = X_{h_N} \circ \phi$. This means that if we can prove that:

$$T_x\phi \cdot X_{h_N \circ \phi}(x) = T_x\phi \cdot X_{h_M}(x) \tag{4.9}$$

then ϕ-relatedness of the control systems follows. Rewriting (4.9) as:

$$\begin{aligned} (\mathcal{L}_{T\phi \cdot X_{h_N \circ \phi}} g)(x) &= (\mathcal{L}_{T\phi \cdot X_{h_M}} g)(x) \qquad \forall g \in C^\infty(N) \\ \Leftrightarrow (\mathrm{d}(f \circ \phi) \cdot X_{h_N \circ \phi})(x) &= (\mathrm{d}(f \circ \phi) \cdot X_{h_M})(x) \\ \Leftrightarrow \{h_N \circ \phi, f \circ \phi\}_M(x) &= \{h_M, f \circ \phi\}_M(x) \\ \Leftrightarrow \{f \circ \phi, h_N \circ \phi\}_M(x) &= \{f \circ \phi, h_M\}_M(x) \\ \Leftrightarrow (\mathrm{d}(h_N \circ \phi - h_M) \cdot X_{f \circ \phi})(x) &= 0 \end{aligned}$$

Since $f \circ \phi$ is constant on the leaves of $\mathcal{K}$ it follows that $X_{f \circ \phi} \in B_M^{\#}(\mathcal{K}^o) \subseteq Ti(TN)$. Therefore $(\mathrm{d}(h_N \circ \phi - h_M) \cdot X_{f \circ \phi})(x) = 0$ since h_N equals h_M on the integral manifold (which is just N) of $Ti(TN)$ passing through the point x. Since the argument does not depend on the point x the conclusion holds for all $x \in M$.

To show that the Hamiltonian control system induced by $\mathcal{H}_N$ is the smallest one ϕ-related to S_{H_M} we will consider any other family of Hamiltonians $\mathcal{G}_N$ inducing a control system ϕ-related to S_{H_M}. This family defines a codistribution on M verifying properties 1 and 2 by $\mathrm{d}(\mathcal{G}_N \circ \phi)$. However since $\mathrm{d}\overline{\mathcal{H}}_M$ is contained in any other such codistribution it follows that $\mathrm{d}\overline{\mathcal{H}}_M \subseteq \mathrm{d}(\mathcal{G}_N \circ \phi)$ which is equivalent to $\phi_B(\mathrm{d}\overline{\mathcal{H}}_M) \subseteq \phi_B(\mathrm{d}\mathcal{G}_N \circ \phi)$ and lead to the desired inclusion $\mathrm{d}\mathcal{H}_N \subseteq \mathrm{d}\mathcal{G}_N$. $\square$

As asserted by Proposition 4.9 the abstraction obtained by the canonical construction is the smallest Hamiltonian control system ϕ-related to S_{H_M}, therefore we are always able to compute the minimal ϕ-abstraction of any Hamiltonian control system given an abstracting Poisson map ϕ.

5. Local Accessibility Equivalence

In addition to propagating trajectories and Hamiltonians from the original Hamiltonian control system to the abstracted Hamiltonian system, we will investigate how accessibility properties can be preserved in the abstraction process. We first review several (local) accessibility properties for control systems [4, 6, 11].

Definition 5.1 (Reachable sets [6])**.** Let S_M be a control system on a smooth manifold M. For each $T > 0$ and each $x \in M$, the set of points reachable from x at time T, denoted by $Reach(x, T)$, is equal to the set of terminal points $c^M(T)$ of S_M trajectories that originate at x. The set of points reachable from x in T or fewer units of time, denoted by $Reach(x, \leq T)$ is given by $Reach(x, \leq T) = \cup_{t \leq T} Reach(x, T)$.

Definition 5.2. A control system S_M is said to be

- Locally accessible from x there is a neighborhood V of x such that $Reach(x, T)$ contains a non-empty open set of M for all $T > 0$ and $Reach(x, T) \subset V$.
- Locally accessible if it is locally accessible from all $x \in M$.
- Controllable if for all $x \in M$, $Reach(x, T) = M$ for some T.

Recall first that local accessibility properties of Hamiltonian control systems can be characterized by simple rank conditions of the Poisson algebra generated by the controlled Hamiltonian.

Proposition 5.3 (Accessibility Rank Conditions [11]). *Let S_{H_M} be a Hamiltonian control system on a Poisson manifold $(M, \{,\}_M)$ of dimension m and denote by $\mathcal{P}(\mathcal{H}_M)$ the Poisson algebra freely generated by the collection of Hamiltonians $\mathcal{H}_M$. Then:*

- *If $dim(\mathrm{d}(\mathcal{P}(\mathcal{H}_M(x)))) = m$, then the control system S_{H_M} is locally accessible at $x \in M$.*
- *If $dim(\mathrm{d}(\mathcal{P}(\mathcal{H}_M(x)))) = m$ for all $x \in M$, then control system S_{H_M} is locally accessible.*
- *If $dim(\mathrm{d}(\mathcal{P}(\mathcal{H}_M(x)))) = m$ for all $x \in M$, $\mathcal{H}_M$ is symmetric, that is $h \in \mathcal{H}_M \Rightarrow -h \in \mathcal{H}_M$, and M is connected, then control system S_{H_M} is controllable.*

Theorem 4.3 immediately propagates local accessibility from the original Hamiltonian system to its abstraction.

Proposition 5.4 (Local Accessibility Propagation). *Let Hamiltonian control systems S_{H_M} and S_{H_N} be ϕ-related with respect to a Poisson map $\phi : M \longrightarrow N$. Then, if S_{H_M} is (symmetrically) locally accessible (at $x \in M$) then S_{H_N} is also (symmetrically) locally accessible (at $\phi(x) \in N$). Also, if S_{H_M} is controllable then S_{H_N} is controllable.*

We now determine under what conditions on the abstracting maps, local accessibility of the original system S_{H_M} is equivalent to local accessibility of its canonical abstraction S_{H_N}. In particular, we need to address the problem of propagating accessibility from the abstracted system S_{H_N} to the original system S_{H_M}. We start by exploring the relationship between the Poisson algebras of canonically ϕ-related Hamiltonian systems.

Lemma 5.5. *Let S_{H_N} be canonically ϕ-related to S_{H_M}, then for all $x \in M$ we have*

$$\phi_B\left(\mathrm{d}\mathcal{P}(\mathcal{H}_M(x))\right) = \mathrm{d}\mathcal{P}(\mathcal{H}_N)(\phi(x))$$

Proof. We start by showing that for any two functions $h_M, h'_M \in C^\infty(M)$ and any point $x \in M$ we have:

$$T_x\phi \cdot X_{\{h_M,h'_M\}_M}(x) = X_{\{h_N,h'_N\}_N}(\phi(x)) \tag{5.1}$$

with $h_N = h_M \circ i$, $h'_N = h'_M \circ i$ and any i embedding N into M through the point x. Since ϕ is Poisson we have that:

$$T_x\phi \cdot X_{\{h_N,h'_N\}_N\circ\phi}(x) = X_{\{h_N,h'_N\}_N}(\phi(x)) \tag{5.2}$$

so that we only have to show that:

$$T_x\phi \cdot X_{\{h_M,h'_M\}_M}(x) = T_x\phi \cdot X_{\{h_N,h'_N\}_N\circ\phi}(x) \tag{5.3}$$

This argument parallels the one in the proof of Proposition 4.9. The previous expression is equivalent to:

$$\begin{aligned}
(\mathcal{L}_{T\phi\cdot X_{\{h_M,h'_M\}_M}} f)(x) &= (\mathcal{L}_{T\phi\cdot X_{\{h_N\circ\phi,h'_N\circ\phi\}_M}} f)(x) \qquad \forall f \in C^\infty(N)\\
\Leftrightarrow \quad \{f\circ\phi, \{h_M, h'_M\}_M\}_M(x) &= \{f\circ\phi, \{h_N\circ\phi, h'_M\circ\phi\}_M\}_M(x)\\
\Leftrightarrow \quad \{f\circ\phi, \{h_M - h_N\circ\phi, h'_M - h'_N\circ\phi\}_M\}_M(x) &= 0\\
\Leftrightarrow \quad -\{h_M - h_N\circ\phi, \{h'_M, h'_N\circ\phi, f\circ\phi\}_M\}_M(x) &\\
-\{h'_M - h'_N\circ\phi, \{f\circ\phi, h_M - h_N\circ\phi\}_M\}_M(x) &= 0 \qquad (5.4)
\end{aligned}$$

But since $\{h'_M - h'_N \circ \phi, f \circ \phi\}_M(x) = (\mathcal{L}_{X_{f\circ\phi}} h'_M - h'_N \circ \phi)(x)$, $X_{f\circ\phi} \in B_M^{\#}(\mathcal{K}^o) \subseteq Ti(TN)$ and $h'_M = h'_N$ on $Ti(TN)$ we conclude that $\{h'_M - h'_N \circ \phi, f \circ \phi\}_M(x) = 0$ and similarly for $\{h_M - h_N \circ \phi, f \circ \phi\}_M(x)$. This shows that equality (5.4) holds, and a induction argument extends (5.1) to:

$$T_x\phi \cdot X_{\mathcal{P}(\mathcal{H}_M)}(x) = X_{\mathcal{P}(\mathcal{H}_N)}(\phi(x)) \tag{5.5}$$

By making use of Proposition 4.4 the above expression is equivalent to:

$$\phi_B\left(\mathrm{d}\mathcal{P}(\mathcal{H}_M(x))\right) = \mathrm{d}\mathcal{P}(\mathcal{H}_N)(\phi(x)) \tag{5.6}$$

□

Using the above lemma, accessibility equivalence between the two control systems can be now asserted.

Theorem 5.6 (Local Accessibility Equivalence). *Let S_{H_N} be canonically ϕ-related to S_{H_M}. If every vector field $K_i \in Ker(T\phi)$ is Hamiltonian with Hamiltonian function $h_i \in C^\infty(M)$ and $h_i \in \mathcal{P}(\mathcal{H}_M)$, then S_{H_M} is locally accessible if and only if S_{H_N} is locally accessible.*

Proof. We begin by showing how accessibility properties of S_{H_M} are propagated to S_{H_N}. Suppose that S_{H_M} is locally accessible, that is $\mathrm{d}\mathcal{P}(\mathcal{H}_M)(x) = T_x^*M$ for all $x \in M$, then by Lemma 5.5 $\mathrm{d}\mathcal{P}(\mathcal{H}_N)(\phi(x)) = \phi_B(x)T_x^*M$. Since $\phi_B = (B_N^{\#})^{-1} \circ T\phi \circ B_M^{\#}$ and both $B_N^{\#}$ and $B_M^{\#}$ are isomorphisms, and $T\phi$ is surjective, ϕ_B is also surjective. We conclude therefore that $\mathrm{d}\mathcal{P}(\mathcal{H}_N)(y) = T_y^*N$, for all $y = \phi(x)$. But ϕ is surjective so S_{H_N} is locally accessible.

Let us now show how accessibility properties of S_{H_N} can be pulled back to S_{H_M}. We proceed by contradiction. Assume that every $K_i \in Ker(T\phi)$ is Hamiltonian with Hamiltonian function $h_i \in \mathcal{P}(\mathcal{H}_M)$ and that S_{H_N} is locally accessible while S_{H_M} is not. Then $\mathrm{d}\mathcal{P}(\mathcal{H}_N)(y) = T_y^*N$ and by Lemma 5.5 $\phi_B \mathrm{d}\mathcal{P}(\mathcal{H}_M)(x) = T_y^*N$ for all x such that $\phi(x) = y$. Since S_{H_M} is not locally accessible there exists some $g \in C^\infty(M)$ such that $\mathrm{d}g(x) \notin \mathrm{d}\mathcal{P}(\mathcal{H}_M)(x)$, but ϕ_B is surjective so $\mathrm{d}g(x)$ must belong to $Ker(\phi_B(x))$. Taking into consideration that $\mathrm{d}g(x) \in Ker(\phi_B(x)) \Leftrightarrow X_g(x) \in Ker(T_x\phi)$ we have a contradiction since we were assuming that all Hamiltonian functions of the vectors belonging to $Ker(T_x\phi)$ were also in $\mathcal{P}(\mathcal{H}_M)(x)$ and $g(x) \notin \mathcal{P}(\mathcal{H}_M)(x)$. This shows that S_{H_M} is in fact locally accessible from x. Since the argument does not depend on the particular point x, S_{H_M} is locally accessible. □

Corollary 5.7. *Let S_{H_N} be canonically ϕ-related to S_{H_M}. If every $K_i \in Ker(T\phi)$ is Hamiltonian with Hamiltonian function $h_i \in C^\infty(M)$, $h_i \in \mathcal{P}(\mathcal{H}_M)$ and both $\mathcal{H}_M$ and $\mathcal{H}_N$ are symmetric and furthermore both M and N are connected then S_{H_M} is controllable iff S_{H_N} is controllable.*

Theorem 5.6 provides moderate conditions to propagate accessibility properties in a hierarchy of abstractions. In fact when dealing with systems affine in controls, that is, of the form $H = H_0 + \sum_i H_i u_i$ we can always build a map ϕ satisfying the conditions of Theorem 5.6 by defining its kernel to be X_{H_i} for some i provided that the conjugate of H_i belongs to the Poisson algebra generated by the control system. A example of this construction is presented in the next section.

6. A spherical pendulum example

As an illustrative example, consider the spherical pendulum as a fully actuated mechanical control system. This system can be used to model, for example, the stabilization of the spinning axis of a satellite or a pan and tilt camera. Consider a massless rigid rod of length l fixed in one end by a spherical joint and having a bulb of mass m on the other end. The configuration space for this control system is S^2 parameterized by $\theta \in [0, \pi[$ and $\phi \in [0, 2\pi[$. The kinetic energy of the system is given by

$$T = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2) \tag{6.1}$$

and the potential energy of the system is

$$V = -mgl\cos\theta \tag{6.2}$$

Trough the Legendre transform of the Lagrangian $L = T - V$ one arrives at the Hamiltonian

$$H_0 = \frac{1}{2ml^2}\, p_\theta^2 + \frac{1}{2ml^2\sin^2\theta}\, p_\phi^2 - mgl\cos\theta \tag{6.3}$$

where p_θ is given by $p_\theta = ml^2\dot{\theta}$ and $p_\phi = ml^2\sin^2\theta\ \dot{\phi}$. Since the system is fully actuated the Hamiltonian control system S_{H_M} defined over $M = T^*S^2$ with the canonical Poisson bracket is given by:

$$H_M = H_0 + H_1 u_1 + H_2 u_2 \tag{6.4}$$

with $H_1 = \theta$ and $H_2 = \phi$ and where u_1 and u_2 are the control inputs.

The drift vector field associated with H_0 is invariant under rotations around the vertical axis and could be reduced using this symmetry. However to emphasize the advantages to the abstraction method we will abstract away precisely the directions were there are no symmetries. Consider the abstracting map:

$$\phi : T^*S^2 \quad \longrightarrow \quad T^*S^1 \tag{6.5}$$

$$(\theta, \phi, p_\theta, p_\phi) \quad \mapsto \quad (\phi, p_\phi) \tag{6.6}$$

It is clear that $\theta \in \mathcal{P}(\mathcal{H}_M)$ and the conjugate variable to θ, p_θ, also belongs to $\mathcal{P}(\mathcal{H}_M)$ since $\{H_0, \theta\}_M = -\frac{1}{ml^2} p_\theta$, so the conditions of Proposition 4.6 are fulfilled meaning that ϕ is a Poisson map inducing a non-degenerate bracket in T^*S^1.

Following the steps of the canonical construction one computes the family of maps:

$$\overline{\mathcal{H}}_M = \mathcal{H}_M \cup \{\theta, \mathcal{H}_M\}_M \cup \{p_\theta, \mathcal{H}_M\}_M \cup \ldots \tag{6.7}$$

However it is enough to compute $\{\theta, H_0\}_M = \frac{1}{ml^2}\, p_\theta$ since $dim(\mathrm{d}(\mathcal{P}(\mathcal{H}_M \cup \{\frac{1}{ml^2}\, p_\theta\}))) = 4$ and all remaining brackets can be generated by $\mathcal{H}_M \cup \{\theta, H_0\}_M$. The collection of Hamiltonians canonically ϕ-related to $\overline{\mathcal{H}}_M$ is given by

$$\mathcal{H}_N = Span\{\frac{1}{2ml^2}\, p_\theta^2 + \frac{1}{2ml^2\sin^2\theta}\, p_\phi^2 - mgl\cos\theta, \theta, \phi, \frac{1}{ml^2}\, p_\theta\} \tag{6.8}$$

but where θ and p_θ are now regarded as control in puts since they are ranging in $\mathcal{K}$. Introducing the new control inputs $v_1 = \theta$ and $v_2 = u_2$ the abstracting controlled Hamiltonian can be written as:

$$H_N = \frac{1}{2ml^2\sin^2 v_1}\, p_\phi^2 + \phi\, v_2 \tag{6.9}$$

Note that the terms depending only on θ or p_θ have disappeared since regarding θ and p_θ as control inputs reduced those terms as constants multiplying the control inputs and constants are associated with the null vector field. The equations of the new control system on $N = T^*S^1$ are obtained through the induced Poisson bracket (which is just the canonical one on N) and are given by:

$$\dot{\phi} = \frac{1}{ml^2 \sin^2 v_1} p_\phi \tag{6.10}$$

$$\dot{p_\phi} = v_2 \tag{6.11}$$

which define a controllable Hamiltonian control system on N.

7. Conclusions

In this paper, we have presented a hierarchical abstraction methodology for Hamiltonian nonlinear control systems. The extra structure of mechanical systems was utilized in to provide constructive methods for generating abstractions while maintaining the Hamiltonian structure. Furthermore we have characterized accessibility equivalence through easily checkable conditions.

These results are very encouraging for hierarchical controlling mechanical systems. Refining controller design from the abstracted to the original system is clearly important. Other research topics under current research include the propagations of nonholonomic constraints among the different levels of the hierarchy, and better understanding the relationship between Hamiltonian abstractions and more established notions of reduction based on symmetries.

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